ON A LOCAL-GLOBAL PRINCIPLE WHICH APPLIES TO MANY GEOMETRIC THEORIES

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ABSTRACT. There is a result known as the fundamental theorem of projective geometry, which says that bijections of projective spaces, with dimension greater than one, over division rings (not necessarily commutative) which preserve collinearity are projective transformations possibly composed with automorphisms of the underlying division ring. The best way to understand this result is as a corollary to the result that the theory of projective spaces in any given dimension greater than one is bi-interpretable with the theory of division rings. This result has many generalizations to various quadratic spaces.

We have obtained versions of these results which only require the mapping to be defined on a certain type of open subset, or on a set of positive measure (for certain classes of topological rings). These results can all be seen as special cases of a "localizability" property that many geometric theories have, which involves the theory being interpretable in itself in a certain way. Alternatively it can be viewed model-theoretically, as a "localizability" property of the models of the theories. We discuss this property and how to derive these results from it.

In this paper, we show how to derive various geometrical theorems from interpretability results. One of the earliest and most famous examples of an interpretability result is the interpretability of hyperbolic geometry in Euclidean geometry, by means of the Beltrami-Klein model. From this result it can be proved in a very weak metatheory that if Euclidean geometry is consistent, then so is hyperbolic geometry. Let us begin by defining the notion of an interpretation of one theory in another.

Our theories will all be either theories in a first-order language, deductively closed in first-order logic, or they will be theories in a second-order language, deductively closed in second-order logic (as it is standardly, and by Gödel incompletely, axiomatized). They may be multi-sorted, that is there may be a finite number, greater than one, of sorts of variables of a given order.

An interpretation of one such language L_1 in another language L_2 of the same order is a mapping which associates to each n-ary nonlogical predicate or function symbol (possibly the equality symbol as well) of L_1, a formula in the language of L_2 whose free variables are $v_1, v_2, \ldots v_{kn}$, for some integer $k > 0$. To each formula ϕ in the language of L_1, we associate the formula in the language of L_2 which is obtained by substituting, for each variable x_i in each occurrence of a predicate or function symbol in ϕ, the variables $v_{k(i-1)+1}, v_{k(i-1)+2}, \ldots v_{ki}$ (which are to be of the same order as x_i, if the language is second-order, and furthermore what sort they are depends only on the sort of x_i) at the corresponding place into the corresponding formula in L_2. To each quantifier in ϕ corresponds a block of k quantifiers of the same kind in the corresponding formula in L_2 in the obvious way. We may further require that each such block of k quantifiers in the formula in L_2 be relativized to some formula expressing a relation between the k variables.

A theory T_1 is said to be interpretable in a theory T_2 if there is an interpretation of the language of T_1 in the language of T_2 which maps theorems of T_1 to theorems of T_2. If the set of theorems of T_1 is mapped onto the set of theorems of T_2, then the interpretation is said to be faithful. Suppose T_i is a theory in the language L_i, $i = 1, 2$, and there exist interpretations $f : L_1 \to L_2, g : L_2 \to L_1$, such that f maps T_1 onto T_2 and g maps T_2 onto T_1, and $g(f(\phi))$ is equivalent to ϕ in T_1 for all formulas $\phi \in L_1$, $f(g(\psi))$ is equivalent to ψ in T_2 for all formulas $\psi \in L_2$. Then the two theories are said to be bi-interpretable.

We shall now describe a family of axiomatizations of n-dimensional projective geometry for each $n > 1$, and show that each theory thereby obtained is bi-interpretable with the theory of division rings. From this we shall deduce the fundamental theorem of projective geometry.

For each integer $n > 1$, we describe an axiomatizable theory PG_n. The language of the theory has just one nonlogical symbol, a ternary predicate symbol $R(x, y, z)$ which is to be interpreted as "x, y and z are collinear." The language also has the equality symbol.

Axiom 1: The relation "x, y and z are collinear" is symmetric in its variables.

Axiom 2: If at least two of x, y and z are equal, then they are collinear.

Axiom 3: If x and y are distinct, and x, y and z are collinear, and x, y and w are collinear, then any three of x, y, z and w are collinear.

Axiom 4: Given two distinct points x and y, there exists a third point z, distinct from x and y, such that x, y and z are collinear.

Axiom 5: There exist three points x, y and z which are not collinear.

We introduce the following family of definitions for each k such that $1 \leq kleqn$, each member of the family building on the previous members of the family.

Definition: We say that x, y, z belong to the same subspace of dimension 1 if they are collinear.

Definition (for each k such that $1 < k \leq n$): We say that $x_1, x_2, \ldots x_{k+2}$ belong to the same subspace of dimension k if some $k + 1$ of them belong to the same subspace of dimension $k-1$, or if there exists a y such that $x_1, x_2, \ldots x_k, y$ belong to the same subspace of dimension $k - 1$, and x_{k+1}, x_{k+2}, y are collinear.

Axiom schema 6 (asserted for each k such that $1 < k < n$): The relation "$x_1, x_2, \ldots x_{k+2}$ belong to the same subspace of dimension k" is symmetric in its variables.

Axiom 7: There exist $n + 1$ points $x_1, x_2, \ldots x_{n+1}$ which do not lie in the same subspace of dimension $n - 1$, and for all y, $x_1, x_2 \ldots x_{n+1}, y$ lie in the same subspace of dimension n.

It is only Axiom 7 which makes the theories PG_n inconsistent with one another. If we let PG_n^* be PG_n minus Axiom 7, then $\{PG_n^* \mid n > 1\}$ is an increasing chain of consistent theories.

Axiom schema 8 (asserted for each k such that $1 \leq k \leq n$): Suppose that $x_1, x_2, \ldots x_{k+1}$ do not belong to the same subspace of dimension $k - 1$ (or, if $k = 1$, that they are distinct). Suppose $y_1, y_2, \ldots y_k$ and $z_1, z_2, \ldots z_k$ are such that for each i such that $1 \leq i \leq k$, $x_1, x_2, \ldots x_{k+1}, y$ belong to the same subspace of dimension k, and the y's do not belong to the same subspace of dimension $k-2$, if $k>2$, or they are distinct if $k=2$, and similarly with the z's. Then there exist $u_1, u_2, \dots u_{k-1}$ which do not belong to the same subspace of dimension $k-2$, if $k>2$, or are distinct if $k=2$, and for each i such that $1 \leq i \leq k-1$, $y_1, y_2, \dots y_k, u_i$ belong to the same subspace of dimension $k-1$, or are equal if $k=1$.

In the case $n=2$, we shall need a further axiom known as Desargues' theorem. For $n>2$, it can be proved as a theorem.

Axiom 9 (Desargues' theorem): Suppose that a_1, a_2 and a_3 are not collinear, and b_1, b_2 and b_3 are not collinear. Then the following two statements are equivalent:

(1) There exist x, y and z such that a_1, a_2, x are collinear, b_1, b_2, x are collinear, a_1, a_3, y are collinear, b_1, b_3, y are collinear, a_2, a_3, z are collinear, b_2, b_3, z are collinear, and x, y, z are collinear.

(2) There exists a point x such that a_1, b_1, x are collinear, a_2, b_2, x are collinear, and a_3, b_3, x are collinear.

As we shall see shortly, the models of PG_n are precisely the n-dimensional projective spaces over division rings. There is a result known as Pappus' theorem, which is not provable in PG_n for any n. It states that if a_1, b_1, c_1 are collinear, a_2, b_2, c_2 are collinear, and whenever $i, j \in \{1, 2, 3\}, i \neq j$, we have $a_i \neq b_j$, and a_1, b_2, x are collinear, a_2, b_1, x are collinear, a_1, b_3, y are collinear, a_3, b_1, y are collinear, a_2, b_3, z are collinear, a_3, b_2, z are collinear, then x, y, z are collinear. If we adjoin Pappus' theorem to PG_n as an axiom, then the models of the resulting theory are precisely the n-dimensional projective spaces over fields. In this sense Pappus' theorem is equivalent to the assertion that multiplication is commutative. The same is true of a generalization of Pappus' theorem, Pascal's theorem. We say that six points in a projective space over a division ring are conconic if they all lie on the same conic section, a conic section being a set S of points such that there exists a symmetric matrix M, such that $x \in S$ if and only if, whenever $\mathbf{x}$ is a homogeneous co-ordinate vector for x, then $\mathbf{x}^T M \mathbf{x} = 0$. It will be seen later that the notion of being conconic is definable in our theory. Pascal's theorem replaces the hypotheses in Pappus' theorem that a_1, b_1, c_1 are collinear and a_2, b_2, c_2 are collinear with the hypothesis that $a_1, b_1, c_1, a_2, b_2, c_2$ are conconic. Pascal's theorem is equivalent to Pappus' theorem in each PG_n, and both are equivalent to the assertion that multiplication is commutative. We shall not be assuming Pappus' theorem as an axiom.

Now we describe the theory of division rings DR. It has the equality symbol. It also has nonlogical constants $0, 1$ and nonlogical binary function symbols $+$ and $\cdot$. We write xy for $x \cdot y$. The first three axioms are

Axiom 1: $0 \neq 1$
Axiom 2: For all x, there exists a y such that $x + y = 0$
Axiom 3: For all x, if $x \neq 0$, there exists a y such that $xy = yx = 1$

The remain axioms are the universal closures of the following formulas.

Axiom 4: $(x + (y + z)) = ((x + y) + z)$
Axiom 5: $x + y = y + x$
Axiom 6: $x + 0 = x$
Axiom 7: $x(yz) = (xy)z$
Axiom 8: $x \cdot 1 = 1 \cdot x = x$

Axiom 9: $x(y+z) = xy + xz$
Axiom 10: $(x+y)z = xz + yz$

The models of this theory are, of course, precisely the division rings.